Lecture 6

Note from last class

$$C(x, \bigtriangledown\phi) = \sum_{|\alpha|=n} A_\alpha \left(\frac{\partial\phi}{\partial x_1}\right)^{\alpha_1} ... \left(\frac{\partial\phi}{\partial x_n}\right)^{\alpha_n} \tag{1}$$

We would like to emphasize that

$$d\phi(x) \in T_x^*M \qquad M - manifold$$

This indicates that C maps tangent bundle $T^*M \mapsto \mathbb{R}$

$$C(x, *) : T_x^*M$$

. The characteristic cone at x is defined by the set

$$\{\xi \in T_x^*M \ : \ C(x, \xi) = 0\}. \tag{2}$$

Lie's Viewpoint

Consider $f(x, y, u, u_x, u_y) = 0$ and some point $p = (x, y, z, t, w) \in \mathbb{R}^5$ then, $f(p) = 0$ is in fact a surface in R^5. This indicates that solving a PDE is essential finding a function $u(x, y)$ that satisfies some surface conditions. In particular

$$J \ : \ u \mapsto (graph\ u, u_x, u_y) \tag{3}$$

$$S \ : \ f(p) = 0 \tag{4}$$

S and J are surfaces in $\mathbb{R}^5$, where we solve for a function u such that

$$Ju \subset S \tag{5}$$

SemiLinear 1^{st} Order equation

Let $\Omega \subset \mathbb{R}^n$ be an Open, Connected and Nonempty (i.e let be a domain). Let $\alpha \ : \ \Omega \mapsto \mathbb{R}^n$ define a vector field and $A := \sum_{i=1}^n \alpha_i(x)\partial_i$ define a differential operator. Then

$$A = \sum_{i=1}^n \alpha_i(x)\partial_i \tag{6}$$

$$\beta \ : \ \Omega \times \mathbb{R} \mapsto \mathbb{R} \tag{7}$$

$$\text{Solve} \qquad Au = \beta(x, u) \tag{8}$$

Definition 1. *Let $I \subset \mathbb{R}$ be an interval.*

$$\gamma \ : \ I \mapsto \mathbb{R}^n$$

is called a Parameterized Characteristic Curve (PC-curve) if

$$\gamma'(t) = \alpha(\gamma(t)) \qquad \forall t \in I$$

C-curve : $[\gamma] = \gamma(I)$ is the image of γ under I.

Class notes by Ibrahim Al Balushi

Lemma 1. *Left* $u: \Omega \mapsto \mathbb{R}$ *be differentiable and let* γ *be a PC-curve, then*

$$\frac{d\ u(\gamma(t))}{dt} = Au\Big|_{\gamma(t)} \qquad t \in I$$

Proof.

$$\begin{aligned} \frac{du(\gamma(t))}{dt} &= \sum_i \partial_i u(\gamma(t))\gamma_i'(t) && (9) \\ &= \sum_i \partial_i u(\gamma(t))\alpha_i(\gamma(t)) && (10) \\ &= (Au)(\gamma(t)) = Au\Big|_{\gamma} && (11) \end{aligned}$$

□

Lemma 2. *Let* $\alpha \in C^1$ *and* u *be differentiable.*

$$Au = \beta(x, u) \ \Leftrightarrow \ \frac{du(\gamma(t))}{dt} = \beta(\gamma(t), u(\gamma(t)))$$

on every PC-curve.

Proof. ($\Rightarrow$) By previous Lemma

$$\frac{du(\gamma(t))}{dt} = (Au)(\gamma(t)) = \beta(\gamma(t), u(\gamma(t)))$$

($\Leftarrow$)
Let $x \in \Omega$. γ PC-curve with $\gamma(0) = x$.

$$\begin{aligned} (Au)(x) &= (Au)(\gamma(t))\Big|_{t=0} && (12) \\ &= \frac{du(\gamma(t))}{dt}\Big|_{t=0} && (13) \\ &\text{by hypothosis} && (14) \\ &= \beta(\gamma(t), u(\gamma(t)))\Big|_{t=0} && (15) \\ &= \beta(x, u(x)) && (16) \end{aligned}$$

□

Cauchy Problem

Consider $\Gamma \subset \Omega$ surface with a function $g: \Gamma \mapsto \mathbb{R}$. Find solution for (8) such that

$$u\Big|_{\Gamma} = g$$

To do this, start with $\xi \in \Gamma$ then solve for PC-curve such that $\gamma(0) = \xi$. Denote $v := u(\gamma(t))$ so v_ξ simply denoted the v which satisfies $\gamma_0 = \xi$. Now Solve for

$$\frac{dv_\xi}{dt} = \beta(\gamma(t), v_\xi(t)) \qquad \text{By Lemma 1} \tag{17}$$

$$\text{with } v_\xi(0) = g(\xi) \tag{18}$$

So the solution for the ODE above $v_\xi(t)$ is a solution for PDE (8) along γ, so set

$$u(\gamma(t)) := v(t)$$

Assume $\alpha \in C^k$ $\beta \in C^k$. We conclude by elementary ODE theory that $\gamma \in C^k$ so

$$f(t, v) := \beta(\gamma(t), v) \qquad f \in C^k$$

We know v is unique (also by elementary ode theory) so if u solves (8) then $u(\gamma(t)) = v_\xi(t)$ by uniqueness.

Assume $\forall \xi \in \Gamma,\ \alpha(\xi)$ is not tangent to Γ and suppose $\Sigma \subset \Gamma \times \mathbb{R}$

$$\underbrace{\varphi \ : \ \Sigma \mapsto \mathbb{R}^n}_{\text{suppose Injective}} \tag{19}$$

$$\varphi(\xi, t) = \gamma_\xi(t) \tag{20}$$

$\implies \varphi :\ \Sigma \mapsto \varphi(\Sigma)$ is bijective. Now suppose we know value of $u(x)$, we can solve for value at ξ by inversion; that is

$$\varphi^{-1}(x) =: (\xi(x), t(x))$$

provided the bijection above holds on a local subset then $u(x) = v_\xi(t(x))$ solves the PDE locally there.

Problem: Is u differentiable?

Inverse Mapping Theorem

Let

$$\varphi \ : A_{\subset \mathbb{R}^n} \mapsto \beta_{\subset \mathbb{R}^n}$$

$D\varphi(x)v := \sum_j v_j \partial_j \varphi(x)$ where D is said to be total differential operator. Suppose $\varphi \in C^k$ and $D\varphi(a)$ invertible

$$\implies \exists X \ni a \ nbhd \ s.t \ \varphi :\ X \mapsto Y = \varphi(X) \in \ C^k$$

I.e we have diffeomorphism, that is,

$$\exists \varphi^{-1} : Y \mapsto X \ and \ \varphi^{-1} \in C^k.$$

In addition, if φ is injective then $\varphi \ : \ A \mapsto B$ is $C^k - diffeomorphism.$

Let

$$\varphi(\xi, t) = \Phi^t(\xi) \qquad \Phi^t \ : \ \Omega \mapsto \mathbb{R}^n$$

$$\Phi^{t+s}(\xi) = \Phi^t(\Phi^s(\xi)) \tag{21}$$

$$\left(\Phi^t\right)^{-1} = \Phi^{-t} \tag{22}$$

$$\frac{\partial}{\partial s}\Phi^{t+s}(\xi) = D\Phi^t(\Phi^s(\xi))\frac{\partial}{\partial s}\Phi^s(\xi) \tag{23}$$

$$s = 0 \ : \ \frac{\partial}{\partial t}\Phi^t(\xi) = D\Phi^t(\xi)\alpha(\xi) \tag{24}$$

$$D\Phi(\xi, t)(\dot{\xi}, \dot{t}) = (D\Phi^t(\xi)\dot{\xi}, D\Phi^t(\xi)\alpha(\xi)\dot{t}) \tag{25}$$

$$\dot{\xi} \in T_\xi\Gamma, \ \alpha(\xi) \notin T_\xi\Gamma \implies D\varphi \ non - degenrated \implies invertible$$